

Optimal decomposable witnesses without the spanning property

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One of the unsolved problems in the characterization of the optimal entanglement witnesses is the existence of optimal witnesses acting on bipartite Hilbert spaces $\mathcal{H}_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^n$ such that the product vectors obeying $\langle e, f | W | e, f \rangle = 0$ do not span $\mathcal{H}_{m,n}$. So far, the only known examples of such witnesses were found among indecomposable witnesses, one of them being the witness corresponding to the Choi map. However, it remains an open question whether decomposable witnesses exist without the property of spanning. Here we answer this question affirmatively, providing systematic examples of such witnesses. Then, we generalize some of the recently obtained results on the characterization of $2 \otimes n$ optimal decomposable witnesses [R. Augusiak *et al.*, *J. Phys. A* **44**, 212001 (2011)] to finite-dimensional Hilbert spaces $\mathcal{H}_{m,n}$ with $m, n \geq 3$.

I. INTRODUCTION

Characterization and classification of entanglement witnesses (EWs) remains an unsolved problem in entanglement theory [1, 2]. Recall that by an entanglement witness [3], we understand a Hermitian operator, usually denoted by W , acting on some bipartite Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$ such that $\langle W \rangle_\sigma \geq 0$ for any separable state [4] σ acting on $\mathbb{C}^m \otimes \mathbb{C}^n$, while for some entangled ϱ , $\langle W \rangle_\varrho < 0$. In the latter case, we say briefly that entanglement of ϱ or just ϱ is detected by the EW W , hence the term [5]. What makes these objects important from an entanglement detection point of view is that (as proven in Refs. [3] and [6]) for any entangled state (bipartite or multipartite) there exists some EW detecting it (i.e., having negative mean value in this state). Much effort has been put toward designing EWs that detect entanglement in various bipartite and multipartite physical systems (see, e.g., Ref. [7]). They also allow for the quantitative analysis of entanglement (see, e.g., Ref. [8]), and, finally, since these objects are just quantum observables, the qualitative and quantitative detection of entanglement is feasible from the experimental point of view [9].

Particularly important for the characterization of EWs is the notion of optimality introduced in Ref. [10] (see also Ref. [11] for a state-dependent definition of optimality). Roughly speaking, optimal EWs are the ones that detect (in the set-theoretic terms) the largest set of entangled states. In other words, a given EW W is optimal if there exist no other witness detecting the same states as W and additionally some states which are not detected by W . As every witness can be optimized [10], it happens that the optimal EWs are sufficient to detect all the entangled states. It is then of significant importance to isolate and characterize the set of optimal EWs.

Many interesting results have been obtained in research aiming to realize this goal (see, e.g., Refs. [12] and [13]). However, though this effort, the characterization of optimal EWs is far from being accomplished, and our knowledge about their structure remains unsatisfactory.

Very recently, in Ref. [14], some of us have have provi-

ded a more exhaustive characterization of all qubit-qunit decomposable EWs (DEWs). It was shown that product vectors orthogonal to any completely entangled subspace (CES) of $\mathbb{C}^2 \otimes \mathbb{C}^n$, after being partially conjugated, span the latter. On the level of DEWs, together with results already established in Ref. [10], this means that for any DEW W acting on $\mathbb{C}^2 \otimes \mathbb{C}^n$ the following two equivalences hold: (i) W is optimal if and only if (iff) it takes the form $W = Q^\Gamma$ with $Q \geq 0$ supported on some CES and (ii) W is optimal iff the qubit-qunit product vectors satisfying $\langle e, f | W | e, f \rangle = 0$ span $\mathbb{C}^2 \otimes \mathbb{C}^n$. Since it is important for our considerations, let us refer to the latter property as *the property of spanning*.

On the other hand, it was shown in Ref. [14] that already the two-qutrit DEWs do not follow the above characterization. More precisely, while for all DEWs $W = Q^\Gamma$ with Q of rank 1 or 2, the above equivalences also hold, in the case when $r(Q) = 3, 4$ either there exist non-optimal DEWs taking the above form or there exist optimal witnesses without the property of spanning. While the former question has very recently been solved [15], the latter one seems particularly interesting and has not been answered so far. The existence of optimal EWs without the property of spanning is already known in the case of indecomposable witnesses, where one has the so-called Choi witness, that is, an indecomposable EW acting on $\mathbb{C}^3 \otimes \mathbb{C}^3$ generated from the known Choi map [16, 17]. (For the proof that this witness does not have the property of spanning, see Ref. [13].) This fact, however, remains unknown in the decomposable case.

The primary purpose of this paper is to clarify this point and show that for any Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$, obviously except for $\mathbb{C}^2 \otimes \mathbb{C}^n$, there always exist such witnesses. The secondary aim is to discuss possible generalizations of the results obtained in Ref. [14] to higher-dimensional DEWs. In particular, we show that under some condition all CESs supported in $\mathbb{C}^n \otimes \mathbb{C}^n$ of dimension $n - 1$ have the property of spanning. Then, we prove that any CES in \mathcal{H}_n of dimension less than $n - 1$ can be extended (i.e., is a subspace) to some $(n - 1)$ -dimensional CES and therefore also inherits this property.

We apply this analysis to the CESs supported in

$\mathbb{C}^4 \otimes \mathbb{C}^4$. We show that all such CESs of dimension 3 obey the above condition. Consequently, for all two-quart DEWs $W = Q^\Gamma$ with $r(Q) \leq 3$, the above equivalences also hold. Together with the already obtained results for two-qubit and two-qutrit DEWs [14], this suggests to conjecture that the above equivalences are valid for DEWs $W = Q^\Gamma$ acting on $\mathbb{C}^n \otimes \mathbb{C}^n$ with $r(Q) \leq n-1$. However, we provide examples of $(n-1)$ -dimensional CESs supported in $\mathbb{C}^m \otimes \mathbb{C}^n$ with $3 \leq m \leq n$ without the property of spanning, meaning that the local dimensions play some role in the above conjecture.

The paper is organized as follows. In Sec. II, we recall basic notions regarding witnesses and optimality. In Sec. III, we present our main results, that is, examples of optimal DEWs without the property of spanning. In Sec. IV, we discuss the general situation with respect to characterization of optimal DEWs. We conclude with Sec. V and outline possible directions for further research.

II. PRELIMINARIES AND THE PROBLEM

First, we set the notation and recall the basic notions and known facts regarding EWs and, in particular, DEWs.

By $\mathcal{H}_{m,n}$, $\mathbb{M}_{m,n}$, and $\mathbb{D}_{m,n}$ we denote, respectively, the product Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$, the set $\mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$ with $\mathbb{M}_d(\mathbb{C})$ standing for the set of $d \times d$ complex matrices, and finally the subset of positive elements of $\mathbb{M}_{m,n}$ with unit trace. In the case when the dimensions of both subsystems are equal, we use shorter notation: \mathcal{H}_m , \mathbb{M}_m , and \mathbb{D}_m , respectively. By the Schmidt rank of a pure state $|\psi\rangle \in \mathcal{H}_{m,n}$ we understand the rank of a density matrix of one of the subsystems of $|\psi\rangle$. Then, we say that a subspace V of $\mathbb{C}^m \otimes \mathbb{C}^n$ is supported in the latter if V cannot be embedded in some Hilbert space $V_1 \otimes V_2$, where either V_1 or V_2 is a proper subspace of \mathbb{C}^m or \mathbb{C}^n , respectively. Finally, we often denote vectors from \mathbb{C}^m in the following way:

$$\mathbb{C}^m \ni |f\rangle = \sum_{i=0}^{m-1} f_i |i\rangle = (f_0, f_1, \dots, f_{m-1}). \quad (1)$$

By an *entanglement witness* (EW), we understand a Hermitian operator $W \in \mathbb{M}_{m,n}$ which is block-positive, i.e., such that $\langle e, f | W | e, f \rangle \geq 0$ holds for any product vector $|e, f\rangle \in \mathcal{H}_{m,n}$, and there exist entangled states ρ for which $\text{Tr}(W\rho) < 0$.

We call an EW W *decomposable* [10] if it can be written as

$$W = P + Q^\Gamma, \quad (2)$$

with P and Q being positive operators. EWs which do not admit this form are called *indecomposable*. From now on we restrict our attention to decomposable EWs since our results concern only this subset.

We can now turn to the notion of optimality. For this purpose, for a given EW $W \in \mathbb{M}_{m,n}$ let us introduce the following sets

$$\mathcal{P}_W = \{|e, f\rangle \in \mathcal{H}_{m,n} | \langle e, f | W | e, f \rangle = 0\} \quad (3)$$

and

$$\mathcal{D}_W = \{\varrho \in \mathbb{D}_{m,n} | \text{Tr}(\varrho W) < 0\}. \quad (4)$$

Now, taking two EWs W_i ($i = 1, 2$), we say that W_1 is *finer* than W_2 if $\mathcal{D}_{W_2} \subseteq \mathcal{D}_{W_1}$. Then, if there does not exist any witness which is finer than $W \in \mathbb{M}_{m,n}$, we call it *optimal*. That is optimal EWs are those that are maximal with respect to the above relation of inclusion.

In Ref. [10], it was shown that an EW $W \in \mathbb{M}_{m,n}$ is optimal if and only if the matrix $\widetilde{W}(\epsilon, P) = W - \epsilon P$ is no longer block-positive for any $\epsilon > 0$ and $P \geq 0$. Clearly, in order to verify the optimality of a given EW W , it suffices to check the above condition for positive P supported on subspaces that are orthogonal to \mathcal{P}_W because for any P for which $PP_W \neq 0$ there always exists a product vector $|e, f\rangle \in \mathcal{P}_W$ such that $\langle e, f | P | e, f \rangle \neq 0$ and therefore $\langle e, f | W - \epsilon P | e, f \rangle = -\langle e, f | P | e, f \rangle < 0$. This fact immediately implies a sufficient condition for optimality:

(i): If \mathcal{P}_W spans $\mathcal{H}_{m,n}$ then the EW $W \in \mathbb{M}_{m,n}$ is optimal.

Since the latter property is directly related to the notion of optimality, let us remember that we refer to it as *property of spanning*.

Furthermore, application of this condition for optimality to the DEWs (2) yields the necessary condition for optimality of DEWs [10]:

(ii): If a given DEW $W \in \mathbb{M}_{m,n}$ is optimal, then it has to be of the form

$$W = Q^\Gamma, \quad Q \geq 0, \quad Q \text{ supported on CES}, \quad (5)$$

that is, Q is a positive operator supported on some completely entangled subspace.

It has been whether the opposite implications of (i) and (ii) also hold. Clearly, a solution to this problem is crucial from the point of view of a complete characterization of optimal DEWs. As already said, both statements become equivalences for all DEWs from $\mathbb{M}_{2,n}$ and some from $\mathbb{M}_{3,3}$ [14]. Quite surprisingly, however, the existence of nonoptimal DEWs taking the form (5) has very recently been reported [15] (see also Ref. [18] in this context), proving that in general the opposite statement to (ii) does not hold. In particular, it is a consequence of the fact that generally one is able to decompose $\mathcal{H}_{m,n}$ (except for $\mathcal{H}_{2,n}$ and \mathcal{H}_3) as a direct sum of two CESs [?] (see, e.g., Ref. [19]). Any such CES can support a DEW of the form (5), however, the latter cannot be tangent to the set of separable states, and thus it is certainly not an optimal witness [15].

The primary aim of the present paper is to disprove also the inverse of (i) for general DEWs. For this purpose, we first prove that some of the two-qutrit witnesses without the property of spanning found in Ref. [14] are optimal. Then, we generalize these examples to every $\mathcal{H}_{m,n}$ with $m, n \geq 3$, showing that every Hilbert space (except for $\mathcal{H}_{2,n}$) admits optimal DEWs (in the sense that there exist DEWs acting on this Hilbert space) without the property of spanning.

It should also be emphasized that our construction provides examples of optimal DEWs without the property of spanning. Recall that in the indecomposable case the only examples known so far arise from the Choi map [16].

Another aim of our paper is to provide some generalizations of the results of Ref. [14] to higher-dimensional Hilbert spaces $\mathcal{H}_{m,n}$.

Let us notice that we can look at the above problems from a more general perspective. To determine the set \mathcal{P}_W for DEWs (5), we need to find the product vectors $|e, f\rangle \in \mathcal{H}_{m,n}$ obeying $0 = \langle e, f | Q^\Gamma | e, f \rangle = \langle e, f^* | Q | e, f^* \rangle$. Because $Q \geq 0$ and $\text{supp}(Q)$ is a CES, the problem of determining \mathcal{P}_W reduces to the problem of determining product vectors in V^\perp for V being a CES of $\mathcal{H}_{m,n}$. Furthermore, it is fairly easy to see that every CES V admits (in the sense of being supported on V) a positive Q giving rise to a proper EW (that is $Q^\Gamma \not\geq 0$). Consequently, instead of working in terms of particular Q s we can work in terms of CESs of $\mathcal{H}_{m,n}$. For any CES V let us then define the analog of \mathcal{P}_W (defined for EWs):

$$\mathcal{P}_V = \{|e, f^*\rangle \in \mathcal{H}_{m,n} \mid |e, f\rangle \in V^\perp\}. \quad (6)$$

Recall that CESs were already investigated in the literature (see Refs. [20–23]), and the largest dimension of a CES in $\mathcal{H}_{m,n}$ is $(m-1)(n-1)$. This translates to an upper bound on the possible ranks of Q in Eq. (5). In the case of $\mathcal{H}_{2,n}$ this reduces to $n-1$ and all CESs of dimension less or equal $n-1$ have the property of spanning [14]. For the higher-dimensional Hilbert spaces, our result clearly establishes an upper bound on the largest dimension for all CESs to have the property of spanning, which is $\dim V < n$. We conjecture that this bound is tight when local dimensions are equal, while we show that when $m < n$ there exist CESs of dimension $n-1$ without the property of spanning.

III. COMPLETELY ENTANGLED SUBSPACES WITHOUT THE PROPERTY OF SPANNING

Let us introduce some additional notation. Clearly, any vector $|x\rangle \in \mathbb{C}^m$ can be written as $|x\rangle = x_0|0\rangle \oplus |\tilde{x}\rangle$ with $|\tilde{x}\rangle \in \text{span}\{|1\rangle, \dots, |m-1\rangle\}$. In what follows, $|\tilde{x}\rangle$ always denotes a vector coming from the above decomposition. Also, for any $A : \mathbb{C}^m \rightarrow \mathbb{C}^m$, by \hat{A} we denote the $(m-1) \times (m-1)$ matrix obtained from A by removing its first row and first column.

A. The case of equal dimensions

Let us start from the case of equal dimensions of both subsystems and consider the m -dimensional subspace V of \mathcal{H}_m spanned by the vectors

$$|\Psi_i\rangle = a_i|0\rangle|i\rangle - b_i|i\rangle|0\rangle \quad (i = 1, \dots, m-1) \quad (7)$$

with $a_i, b_i \neq 0$ and $|\Psi_m\rangle$ being so far any non-product vector from $(\text{span}\{|1\rangle, \dots, |m-1\rangle\})^{\otimes 2}$. It is useful to notice that via the vectors-matrices isomorphism (see the Appendix), $|\Psi_i\rangle$ ($i = 1, \dots, m-1$) correspond to matrices $A_i = a_i|0\rangle\langle i| - b_i|i\rangle\langle 0|$ and $|\Psi_m\rangle$ correspond to some, so far unspecified, matrix A_m having the entries of the first row and the first column all equal to zero and whose rank obeys $2 \leq r(A_m) \leq m-1$. For the sake of simplicity we assume that $r(A_m) = m-1$ ($|\Psi_m\rangle$ has Schmidt rank $m-1$).

Before going into detail, it should be noticed that by a local nonsingular transformation on one of the subsystems, the subspace V can be transformed to a subspace spanned by $|\Psi'_i\rangle = |0\rangle|i\rangle - |i\rangle|0\rangle$ ($i = 1, \dots, m-1$) and $|\Psi'_m\rangle$ which is again a nonproduct vector from $(\text{span}\{|1\rangle, \dots, |m-1\rangle\})^{\otimes 2}$. Since local nonsingular transformations do not influence our analysis at all, in further considerations we can assume that $a_i = b_i = 1$.

Finally, it is fairly easy to see that V is a completely entangled subspace (see the Appendix for the detailed proof).

Now, let us find all the product vectors $|x, y\rangle$ ($|x\rangle, |y\rangle \in \mathbb{C}^m$) orthogonal to V . This can be done by solving a system of m linear homogenous equations

$$\langle \Psi_i | x, y \rangle = 0 \quad (i = 1, \dots, m), \quad (8)$$

which can be restated as the system of equations for the vector $|y\rangle$ of the form

$$M(x)|y\rangle = 0 \quad (9)$$

with the $m \times m$ $|x\rangle$ -dependent matrix $M(x)$ given by

$$M(x) = \begin{pmatrix} x_1 & -x_0 & 0 & \dots & 0 \\ x_2 & 0 & -x_0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_{m-1} & 0 & \dots & \dots & -x_0 \\ 0 & \langle x^* | A_m^* | 1 \rangle & \dots & \dots & \langle x^* | A_m^* | m-1 \rangle \end{pmatrix}. \quad (10)$$

Clearly, Eq. (9) has a nontrivial solution iff $\det M(x) = 0$. After simple algebra the latter can be rewritten as

$$\begin{aligned} \det M(x) &= (-1)^m x_0^{m-1} \sum_{i,j=1}^{m-1} x_i [A_m^*]_{ij} x_j \\ &= (-1)^m x_0^{m-1} \langle x^* | A_m^* | x \rangle \\ &= 0, \end{aligned} \quad (11)$$

and is satisfied iff either $x_0 = 0$ or $\langle x^* | A_m^* | x \rangle = 0$. Both conditions induce two sets of product vectors in V^\perp , denoted henceforth as \mathcal{S}_1 and \mathcal{S}_2 .

In the first case, when $x_0 = 0$, it follows from Eqs. (9) and (10) the vector $|y\rangle$ has to obey two conditions $y_0 = 0$ and $\langle x^*|A_m^*|y\rangle = \langle \tilde{x}^*|\tilde{A}_m^*|\tilde{y}\rangle = 0$. Consequently, the set \mathcal{S}_1 consists of product vectors taking the form

$$(0, |\tilde{x}\rangle) \otimes (0, (\tilde{A}_m^T|\tilde{x}^*\rangle)^\perp) \quad (12)$$

with arbitrary $|\tilde{x}\rangle$ and $(\tilde{A}_m^T|\tilde{x}^*\rangle)^\perp$ denoting any vector from $(m-2)$ -dimensional subspace of \mathbb{C}^{m-1} orthogonal to $\tilde{A}_m^T|\tilde{x}^*\rangle$.

In the second case, when $\langle x^*|A_m^*|x\rangle = \langle \tilde{x}^*|\tilde{A}_m^*|\tilde{x}\rangle = 0$, it clearly follows from (10) that $|y\rangle = (y_0/x_0)|x\rangle$ (we can obviously assume $x_0 \neq 0$), which together with the above condition determines the second set \mathcal{S}_2 of product vectors in V^\perp . These are precisely the vectors taking the following form:

$$(x_0, |\tilde{x}\rangle) \otimes (x_0, |\tilde{x}\rangle) \quad (13)$$

with $|\tilde{x}\rangle$ obeying $\langle \tilde{x}^*|\tilde{A}_m^*|\tilde{x}\rangle = 0$.

We are now prepared to show that for an appropriate choice of the matrix A_m , the set \mathcal{P}_V does not span \mathcal{H}_m . For this purpose, we denote by \mathcal{S}_1^* and \mathcal{S}_2^* the partially conjugated vectors from \mathcal{S}_1 and \mathcal{S}_2 , respectively. Then, clearly, $\mathcal{P}_V = \mathcal{S}_1^* \cup \mathcal{S}_2^*$.

Since we have still some freedom in the vector $|\Psi_m\rangle$, let us check under which conditions each set does not span \mathcal{H}_m . First, it is fairly straightforward to see that $\dim \text{span} \mathcal{S}_1^* = (m-1)^2$. For this purpose let us determine the conditions under which a vector $|\psi\rangle = \sum_{ij} \alpha_{ij} |ij\rangle \in \mathcal{H}_m$ is orthogonal to all the elements of \mathcal{S}_1^* .

At the beginning, observe that due to the form of vectors from \mathcal{S}_1 , as with one of the indices being zero are arbitrary, implying already that $\dim \text{span} \mathcal{S}_1^* \leq (m-1)^2$. In order to prove equality, we show that the remaining as, forming a matrix $\tilde{\alpha}$ (see the Appendix), have to be 0. In terms of $\tilde{\alpha}$, the condition for $|\psi\rangle$ to be orthogonal to \mathcal{S}_1^* , takes the form

$$\langle \tilde{x}|\tilde{\alpha}|\tilde{y}\rangle = 0 \quad (14)$$

for any $|\tilde{x}\rangle, |\tilde{y}\rangle \in \mathbb{C}^{m-1}$ obeying $\langle \tilde{x}^*|\tilde{A}_m^*|\tilde{y}\rangle = 0$. This is equivalent to saying that $|\tilde{y}\rangle \perp \tilde{\alpha}^\dagger|\tilde{x}\rangle$ for an arbitrary $(m-1)$ -dimensional $|\tilde{x}\rangle$ and $|\tilde{y}\rangle$ fulfilling $|\tilde{y}\rangle \perp \tilde{A}_m^T|\tilde{x}^*\rangle$. By the assumption that \tilde{A} is of full rank, $\tilde{A}_m^T|\tilde{x}^*\rangle \neq 0$ for any $|\tilde{x}^*\rangle$. Then, for a fixed $|\tilde{x}\rangle$, the vector $\tilde{\alpha}^\dagger|\tilde{x}\rangle$ is perpendicular to the $(m-2)$ -dimensional subspace $(\tilde{A}_m^T|\tilde{x}^*\rangle)^\perp$ in \mathbb{C}^{m-1} , so $\tilde{\alpha}^\dagger|\tilde{x}\rangle = k_x \tilde{A}_m^T|\tilde{x}^*\rangle$, where k_x is a scalar depending on $|\tilde{x}\rangle$.

Now, one easily checks that for all $|\tilde{x}\rangle$, the constant k_x has to be independent of $|\tilde{x}\rangle$; let it be denoted by k . Consequently, the condition that $\tilde{\alpha}$ must obey may be now written as $\tilde{\alpha}^\dagger|\tilde{x}\rangle = k \tilde{A}_m^T|\tilde{x}^*\rangle$ for all $|\tilde{x}\rangle \in \mathbb{C}^{m-1}$. Let us write the latter for two vectors, $|\tilde{x}\rangle$ and $i|\tilde{x}\rangle$, which gives us

$$\begin{aligned} \tilde{\alpha}^\dagger|\tilde{x}\rangle &= k \tilde{A}_m^T|\tilde{x}^*\rangle, \\ i\tilde{\alpha}^\dagger|\tilde{x}\rangle &= -ik \tilde{A}_m^T|\tilde{x}^*\rangle. \end{aligned}$$

It means that for any $|\tilde{x}\rangle \in \mathbb{C}^{m-1}$, $\tilde{\alpha}^\dagger|\tilde{x}\rangle = -\tilde{\alpha}^\dagger|\tilde{x}\rangle$, and hence $\tilde{\alpha} = 0$.

Second, the dimension of $\text{span} \mathcal{S}_2^*$ depends on the matrix A_m . This is because one of the conditions defining \mathcal{S}_2^* that $|\tilde{x}^*\rangle \perp \tilde{A}_m^T|\tilde{x}\rangle$ can be seen as a second-order equation for some x_i ($i = 1, \dots, m-1$). In the case when the latter has two different solutions, x_i is a nonlinear function of the remaining x_j ($j \neq i$) (and also of the entries of A_m) and thus linearly independent of them. In such case \mathcal{S}_2^* spans \mathcal{H}_m .

It may happen, however, that $\langle \tilde{x}^*|\tilde{A}_m^*|\tilde{x}\rangle = 0$ has a single solution with respect to some x_i ($i = 1, \dots, m-1$) and then it is just a linear combination of the remaining entries of $|\tilde{x}\rangle$. As a result, the set \mathcal{S}_2^* spans only $(m-1)^2$ -dimensional subspace. This is the situation we look for because it may lead (and, as we will see shortly, does lead) to optimal witnesses without the property of spanning. For instance, this happens when \tilde{A}_m is given by

$$\tilde{A}_m = \begin{pmatrix} 1 & 2 & \dots & 2 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \quad (15)$$

In this case, a direct check shows that one of the conditions defining the set \mathcal{S}_2 reduces now to

$$\langle \tilde{x}^*|\tilde{A}_m^*|\tilde{x}\rangle = x_1 + \dots + x_{m-1} = 0. \quad (16)$$

From now on, let us concentrate on this particular case [i.e., when \tilde{A}_m is given by Eq. (15)] and show that the completely entangled subspace V defined in this way does not have the property of spanning, that is, \mathcal{P}_V does not span \mathcal{H}_m . To this end, we determine $(\text{span} \mathcal{P}_V)^\perp$. Let us take again the vector $|\psi\rangle = \sum_{i,j=0}^{m-1} \alpha_{ij} |ij\rangle$ from \mathcal{H}_m . We already know that it is orthogonal to \mathcal{S}_1^* iff $\alpha_{ij} = 0$ for $i, j = 1, \dots, m-1$. Then, the resulting $|\psi\rangle$ is orthogonal to \mathcal{S}_2^* iff $\alpha_{00} = 0$, $\alpha_{0i} = \alpha_{0,m-1}$, and $\alpha_{i0} = \alpha_{m-1,0}$ with $i = 1, \dots, m-2$. As a consequence, the orthogonal complement of $\text{span} \mathcal{P}_V$ is a two-dimensional subspace of \mathcal{H}_m spanned by the vectors

$$|\phi_1\rangle = |0\rangle|\omega\rangle, \quad |\phi_2\rangle = |\omega\rangle|0\rangle, \quad (17)$$

where $|\omega\rangle = |1\rangle + \dots + |m-1\rangle$.

B. The case of arbitrary dimensions

Before going into detail, let us comment on the notation we use throughout this section. Given a vector $|z\rangle$ from \mathbb{C}^n , we often decompose it as $|z\rangle = |z'\rangle \oplus |z''\rangle$ with $|z'\rangle \in \text{span}\{|0\rangle, \dots, |m-1\rangle\}$ and the rest $|z''\rangle$.

In this section, we extend the already defined m -dimensional CES V of \mathcal{H}_m to an n -dimensional CES \bar{V} in the arbitrary dimensional Hilbert space $\mathcal{H}_{m,n}$ with $m, n \geq 3$. For simplicity, let us assume that $n > m$. We

prove that the subspace \overline{V} does not have the property of spanning.

First, let us take the vectors $|\Psi_i\rangle$ ($i = 1, \dots, m$) (with $|\Psi_m\rangle$ defined in Eq. (15), embed them in $\mathbb{C}^m \otimes \mathbb{C}^n$ ($n > m$), and then supply with the following $m - n$ vectors:

$$|\Psi_i\rangle = |1\rangle|i-2\rangle - |2\rangle|i-1\rangle \quad (i = m+1, \dots, n). \quad (18)$$

Clearly, $\overline{V} = \text{span}\{|\Psi_i\rangle\}_{i=1}^n$ is a n -dimensional subspace in $\mathcal{H}_{m,n}$, which does not contain any product vector (see the Appendix for the detailed proof).

As before, looking for the product vectors $|x, y\rangle$ ($|x\rangle \in \mathbb{C}^m, |y\rangle \in \mathbb{C}^n$) orthogonal to \overline{V} , we arrive at the system of linear equations [similar to the one given in Eq. 8], which can be stated as

$$N(x)|y\rangle = 0, \quad (19)$$

with the $n \times n$ matrix $N(x)$ given by

$$N(x) = \left(\begin{array}{c|cccc} & M(x) & & & \\ \hline & & \mathbf{0}_{n-m} & & \\ \hline 0 & \dots & 0 & x_0 & -x_1 & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & x_0 & -x_1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 0 & x_0 & -x_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & x_0 & -x_1 \end{array} \right), \quad (20)$$

$M(x)$ being the $m \times m$ matrix already introduced in Eq. (10), and $\mathbf{0}_{n-m}$ being the $(n-m) \times (n-m)$ zero matrix.

The above system has nontrivial solutions iff $\det N(x) = 0$, meaning that either $\det M(x) = 0$ or $x_1 = 0$. We already know that the first condition holds iff either $x_0 = 0$ or $x_1 + \dots + x_{m-1} = 0$, giving us, as before, two sets of product vectors orthogonal to \overline{V} (denoted respectively by $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$). Additionally, however, we have a third set of vectors $\overline{\mathcal{S}}_3$ corresponding to the third solution $x_1 = 0$ of the above determinantal equation. Let us now determine these sets.

The first one, $\overline{\mathcal{S}}_1$, we get by putting $x_0 = 0$ in Eq. (20). In this case, the matrix $N(x)$ has a simple block-diagonal form, that is, $N(x) = M(x) \oplus (-x_1)\mathbb{1}_{n-m}$, with $\mathbb{1}_d$ denoting the $d \times d$ identity matrix. Clearly, Eq. (19) implies that $|y\rangle$ has to satisfy $-x_1|y''\rangle = 0$ and $M(x)|y'\rangle = 0$. While the former condition immediately gives $|y''\rangle = 0$, the latter one, together with $x_0 = 0$, implies as before that $y_0 = 0$ and that $|y'\rangle$ has to be orthogonal to $A_m^T|x^*\rangle$ or, in other words, has to obey the equation

$$\sum_{i=1}^{m-1} x_i y_i + 2 \sum_{i < j=1}^{m-1} x_j y_k = 0. \quad (21)$$

In conclusion, $\overline{\mathcal{S}}_1$ consists of vectors

$$(0, x_1, \dots, x_{m-1}) \otimes [(0, y_1, \dots, y_{m-1}) \oplus 0_{n-m}], \quad (22)$$

where all x s and y s have to obey the condition (21) and 0_{n-m} stands for $(n-m)$ -dimensional zero vector. Let us

notice that this is the same class as \mathcal{S}_1 (see Sec. III A) but is embedded in a larger Hilbert space $\mathcal{H}_{m,n}$.

In the second case, when $x_1 + \dots + x_{m-1} = 0$, Eqs. (19) and (20) imply that $M(x)|y'\rangle = 0$ and $y_i = (x_0/x_1)y_{i-1}$ for $i = m, \dots, n-1$. From the first condition we simply get $|y'\rangle = (y_0/x_0)|x\rangle$. The remaining equations can be rewritten as $y_i = (x_0/x_1)^{i-m+1}y_{m-1}$ and then, taking into account that $y_{m-1} = (y_0/x_0)x_{m-1}$, as $y_i = (y_0/x_0)(x_0/x_1)^{i-m+1}x_{m-1}$ with $i = m, \dots, n-1$. Consequently, the second set $\overline{\mathcal{S}}_3$ has the following elements:

$$|x\rangle \otimes [|x\rangle \oplus x_{m-1}(t, t^2, \dots, t^{n-m})], \quad (23)$$

where $t = y_0/x_0$ and x s have to satisfy $x_1 + \dots + x_{m-1} = 0$.

The third set $\overline{\mathcal{S}}_1$ consists of product vectors solving the system (19) in the case when $x_1 = 0$. From Eqs. (19) and (20), it follows that $|y\rangle$ has to obey $M(x)|y'\rangle = 0$ and $x_0 y_i = 0$ for $i = m-1, \dots, n-2$. While the latter conditions immediately imply that $y_i = 0$ for $i = m-1, \dots, n-2$, the former, together with the initial condition $x_1 = 0$, yields $y_1 = 0$, $|y'\rangle = (y_0/x_0)|x\rangle$, and $x_2 + \dots + x_{m-2} = 0$. Taking all these conditions together, we see that the product vectors from $\overline{\mathcal{S}}_1$ take the form

$$|x\rangle \otimes [(y_0/x_0)|x\rangle \oplus (0, \dots, 0, y_{n-1})], \quad (24)$$

with $|x\rangle$ given by

$$|x\rangle = (x_0, 0, x_2, \dots, x_{m-2}, 0), \quad (25)$$

where $x_2 + \dots + x_{m-2} = 0$.

Having determined all the product vectors in V^\perp , we can now show that \mathcal{P}_V does not span $\mathcal{H}_{m,n}$. Recall that \mathcal{P}_V consists of partial conjugations of elements of the sets \mathcal{S}_i . Denoting partially conjugated vectors from $\overline{\mathcal{S}}_i$ by $\overline{\mathcal{S}}_i^*$ ($i = 1, 2, 3$), we have $\mathcal{P}_V = \overline{\mathcal{S}}_1^* \cup \overline{\mathcal{S}}_2^* \cup \overline{\mathcal{S}}_3^*$.

First, let us notice that $\text{span } \mathcal{P}_V = \text{span } \overline{\mathcal{S}}_1^* \cup \overline{\mathcal{S}}_2^*$. Second, let us take a vector $|\psi\rangle \in \mathcal{H}_{m,n}$ with entries in the standard basis denoted by α_{ij} . Short algebra shows that it is orthogonal to $\overline{\mathcal{S}}_1^*$ iff $\alpha_{ij} = 0$ ($i, j = 1, \dots, m-1$). Then, $|\psi\rangle$ is orthogonal to $\overline{\mathcal{S}}_2^*$ iff the following set of condition holds

$$\begin{aligned} \alpha_{00} &= 0, \\ \alpha_{0j} &= 0 \quad (j = m, \dots, n-1), \\ \alpha_{i0} &= \alpha_{i0} \quad (i = 2, \dots, m-1), \\ \alpha_{0j} &= \alpha_{01} \quad (j = 2, \dots, m-1), \\ \alpha_{ij} &= \alpha_{1j} \quad (i = 2, \dots, m-1; j = m, \dots, n-1). \end{aligned} \quad (26)$$

From this analysis, it clearly follows that \mathcal{P}_V does not span $\mathcal{H}_{m,n}$ and the $(n-m+2)$ -dimensional subspace $K = (\text{span } \mathcal{P}_V)^\perp$ is spanned by the vectors $|\phi_1\rangle, |\phi_2\rangle$, and $|\phi_{j-m+3}\rangle = |\omega\rangle|j\rangle$ ($j = m, \dots, n-1$).

C. Optimal decomposable witnesses

What remains to be proven is that the subspace \overline{V} admits optimal decomposable witnesses, that is, one is

able to find a positive Q supported on \bar{V} of rank n such that Q^Γ is a optimal decomposable EW. As we will see shortly, this can be achieved by taking Q of the form

$$Q = \sum_{i=1}^n \lambda_i |\Psi_i\rangle\langle\Psi_i| \quad (27)$$

with $\lambda_i > 0$ ($i = 1, \dots, n$). Clearly, by definition, Q is a positive matrix supported on \bar{V} of rank n , and moreover, it is straightforward to see that it is NPT for any choice of λ s, thus giving rise to proper DEWs. Notice also that here \mathcal{P}_W of $W = Q^\Gamma$ is exactly the same as $\mathcal{P}_{\bar{V}}$.

According to what was said in Sec. II, we need to prove that the operator $\bar{W}(\epsilon, P) = Q^\Gamma - \epsilon P$ is no longer a witness for any $\epsilon > 0$, with P being positive matrices obeying $P \perp \mathcal{P}_{\bar{V}}$. The latter are the positive operators supported on $(n - m + 2)$ -dimensional subspace $\mathcal{K} = (\text{span}\mathcal{P}_V)^\perp = \text{span}\{|\phi_i\rangle\}_{i=1}^{n-m+2}$. Let us prove our statement for the case when P is just a one-dimensional projector onto a general vector from \mathcal{K} , that is,

$$|\varphi\rangle = \sum_{i=1}^{n-m+2} a_i |\phi_i\rangle \quad (a_i \in \mathbb{C}). \quad (28)$$

Then, clearly, the proof will follow for an arbitrary $P \geq 0$ supported on \mathcal{K} .

Denoting by P_φ the projector onto $|\varphi\rangle$, one checks that for the product vector $|u\rangle \otimes |v^*\rangle$, where $|v\rangle = |u\rangle \oplus u_0 1_{n-m}$ and $u_0 = u_1 = u_{m-1}$, the following holds:

$$\begin{aligned} \langle u, v^* | Q^\Gamma - \epsilon P_\varphi | u, v^* \rangle &= \langle u, v | Q | u, v \rangle - \epsilon |\langle \varphi | u, v^* \rangle|^2 \\ &= \lambda_m |\langle \Psi_m | u, u \rangle|^2 - \epsilon |\langle \varphi | u, v^* \rangle|^2 \\ &= \lambda_m |\bar{u}|^4 - \epsilon |a_1 u_0 \bar{u}^* + (a_2 + a_3 + \dots + a_{n-m+2}) u_0^* \bar{u}|^2, \end{aligned} \quad (29)$$

where we denoted $\bar{u} = u_1 + \dots + u_{m-1}$. The second equality is a consequence of the fact that $|u\rangle|v\rangle$ is orthogonal to $|\Psi_i\rangle$ ($i = 1, \dots, m-1$) and $|\Phi_i\rangle$ ($i = m, \dots, n-1$), while the third one stems from the fact that $\langle \Psi_m | u, u \rangle = (u_1 + \dots + u_{m-1})^2 = \bar{u}^2$.

For simplicity, let us now put $u_0 = 1$ and denote $a = a_2 + \dots + a_{n-m+2}$. Then, Eq. (29) can be rewritten as

$$\begin{aligned} \langle u, v^* | Q^\Gamma - \epsilon P_\varphi | u, v^* \rangle &= \lambda_m |\eta|^4 - \epsilon |a_1 \eta + a \eta^*|^2 \\ &= |\eta|^2 \left(\lambda_m |\eta|^2 - \epsilon |a_1 + a e^{-2i\delta}|^2 \right), \end{aligned} \quad (30)$$

where the second equality follows from the fact that we can always write $\eta = |\eta|e^{i\delta}$. Clearly, we can always choose δ such that $|a_1 + a e^{-2i\delta}| > 0$ and then, taking sufficiently small $|\eta|$, we can make $\langle u, v^* | Q^\Gamma - \epsilon P_\varphi | u, v^* \rangle$ negative for any $\epsilon > 0$.

Eventually, let us observe that the same reasoning applied when P is any positive matrix supported on \mathcal{K} .

Concluding Q given by Eq. (27) gives rise to an optimal decomposable witness such that the corresponding P_W does not span $\mathcal{H}_{m,n}$.

IV. FURTHER RESULTS ON THE GENERAL OPTIMAL DECOMPOSABLE WITNESSES

Let us now ask whether and how the results obtained in Ref. [14] for qubit-qunit witnesses can be generalized to DEWs acting on $\mathcal{H}_{m,n}$ with $m, n \geq 3$. Following Ref. [14] and the structure of equations defining product vectors in an orthogonal complement of some CES, we surmise that for any CES V of $\mathcal{H}_{m,n}$ with $\dim V \leq n-1$ the corresponding \mathcal{P}_V spans $\mathcal{H}_{m,n}$. The main purpose of this section is to comment on this issue.

First, we show that under some assumption, which is generically obeyed and conjectured to hold always, all CESs of \mathcal{H}_n of dimension $n-1$ have the property of spanning. Then, we prove that any r -dimensional CES is a subspace of some $(r+1)$ -dimensional CES. Consequently, provided the above conjecture holds, all CESs in \mathcal{H}_n of dimension $n-1$ or less have the property of spanning.

Then, we prove that the mentioned assumption is always fulfilled for CESs supported in \mathcal{H}_4 of dimension 3. Therefore, due to the above fact, any CES supported in \mathcal{H}_4 of dimension less or equal to 3 has the property of spanning, and hence for all witnesses (5) in \mathbb{M}_4 with $r(Q) \leq 3$, the statements (i) and (ii) (see Sec. II) become equivalences.

To be more precise, let $V \subset \mathcal{H}_n$ be an $(n-1)$ -dimensional CES spanned by $|\Psi_i\rangle$ ($i = 1, \dots, k$). Product vectors in V^\perp must obey the set of equations $\langle \Psi_i | x, y \rangle = 0$ ($i = 1, \dots, n-1$), which, as we already know, can be rewritten as

$$B(x)|y\rangle = 0, \quad (31)$$

with the $|x\rangle$ -dependent $(n-1) \times n$ matrix $B(x)$. For further benefit, let us denote by Π_V the projector onto V and by $\Pi_V(x)$ a local projection of Π_V onto $|x\rangle \in \mathbb{C}^n$, that is, $\Pi_V(x) = \text{Tr}[(|x\rangle\langle x| \otimes \mathbb{1})\Pi_V]$. Analogously, $|\Psi(x)\rangle$ stands for the local projection of some composite vector $|\Psi\rangle \in \mathcal{H}_n$ onto $|x\rangle \in \mathbb{C}^n$. In both cases, for concreteness, we always project at the first subsystem.

Now we are prepared to prove the following theorem.

Theorem 1. *Consider an $(n-1)$ -dimensional subspace $V \subset \mathcal{H}_n$ possessing the following properties:*

1. *Local projection of Π_V onto at least one vector $|x\rangle \in \mathbb{C}^n$ gives a full-rank density matrix of the other subsystem.*
2. *The subspace V cannot be embedded in any $V_1 \otimes V_2$, where V_i or V_2 is a proper subspace of \mathbb{C}^n .*

Then \mathcal{P}_V spans the whole \mathcal{H}_n .

Proof. Assume that V is spanned by the the following orthonormal vectors:

$$|\Psi_i\rangle = \sum_{k,l=0}^{n-1} a_{kl}^{(i)} |k\rangle|l\rangle \quad (i = 0, \dots, n-1). \quad (32)$$

As before, A_i ($i = 0, \dots, n-1$) are square matrices corresponding to $|\Psi_i\rangle$, that is, matrices formed from the numbers $a_{kl}^{(i)}$.

Product vectors in V^\perp are solutions to the system of equations (31). Assume now that for some $|x\rangle \in \mathbb{C}^n$ the matrix $B(x)$ is of full rank, and denote by $M_i(x)$ ($n = 0, \dots, n-1$) the matrices obtained by removing the i th column from $B(x)$. Then this system has a unique solution (and the corresponding product vector in V^\perp) given by $|y(x)\rangle = [y_0(x), \dots, y_{n-1}(x)]$ with $y_i(x)$ being the determinant of $M_i(x)$, that is,

$$y_i(x) = \det M_i(x) \quad (i = 0, \dots, n-1). \quad (33)$$

Now, let us notice that $B(x)$ is of full rank iff $\Pi_V(x)$ is. This is because

$$\Pi_V(x) = \sum_i |\Psi_i(x)\rangle\langle\Psi_i(x)|, \quad (34)$$

and, clearly, $\Pi_V(x)$ is of full rank iff all the vectors $|\Psi_i(x)\rangle$ ($i = 1, \dots, n-1$) are linearly independent. Since $|\Psi_i(x)\rangle$ are just rows of $B(x)$, we infer that $B(x)$ is of full rank iff $\Pi_V(x)$ is.

The first assumption, together with the above fact, tell us that there exists a vector $|x\rangle \in \mathbb{C}^n$ such that $B(x)$ is of full rank. Let $|x_0\rangle$ denote such a vector. Consequently one of the matrices $M_i(x)$ has nonvanishing determinant at $|x_0\rangle$. On the other hand, by the very definition, $y_i(x)$ is a homogenous polynomial in coordinates of $|x\rangle$. Since then $y_i(x)$ does not vanish at at least one x , it cannot be identically equal zero. As, moreover, the equation $y_i(x) = 0$ has solutions only in the set of Lebesgue's measure zero, the matrix $B(x)$ is of full rank for almost all $|x\rangle \in \mathbb{C}^n$. In other words, the system (31) has a unique solution for almost all $|x\rangle$ s given by Eqs. (33). For the remaining $|x\rangle$ s, it has at least a two-dimensional subspace of solutions; however, in such cases Eqs. (33) give us a zero vector. Nevertheless, the set of product vectors $|x\rangle|y(x)\rangle$ with $|y(x)\rangle$ given by Eqs. (33) is enough to span, after being partially conjugated, the Hilbert space \mathcal{H}_n . For convenience, let us denote these solutions by \mathcal{C} and their partial conjugations by \mathcal{C}^* .

Notice that the above reasoning holds when we exchange the subsystems, that is, for almost all $|y\rangle \in \mathbb{C}^n$ there exist $|x\rangle \in \mathbb{C}^n$ such that y is the unique (up to a scalar) solution of Eq. (31).

Because the polynomial formulas (33) for coordinates of the solution of Eq. (31) produce almost all vectors $|y\rangle$ from \mathbb{C}^n , the polynomials (33) are linearly independent.

It remains to prove that the vectors from \mathcal{C}^* , (i.e., vectors $|x\rangle \otimes |y(x^*)\rangle$) span $\mathbb{C}^n \otimes \mathbb{C}^n$. To this end, observe that $|y\rangle$ is defined by n linearly independent polynomials of variables x_0^*, \dots, x_{n-1}^* . Moreover, they are linearly independent of the polynomials x_0, \dots, x_{n-1} (no x_i^* can be achieved by a combination of x_i s so also no y_i^*). It implies, that the coordinates of $|x\rangle \otimes |y(x^*)\rangle$ form a set of n^2 linearly independent polynomials, so the set $\{|x\rangle \otimes |y(x^*)\rangle : |x\rangle \in \mathbb{C}^n \wedge \text{rank} B(x) = n-1\}$ spans

the whole $\mathbb{C}^n \otimes \mathbb{C}^n$. \square

Remark 1. Any subspace W in $\mathbb{C}^n \otimes \mathbb{C}^n$ of dimension less than $n-1$, which can be extended to $(n-1)$ -dimensional subspace (which we show to be the case in Lemma 2), satisfying the assumptions of the theorem, inherits the property of spanning from V . Indeed, all product vectors orthogonal to V are orthogonal to W , so already a subset of product vectors orthogonal to W spans the whole $\mathbb{C}^n \otimes \mathbb{C}^n$ after partial conjugation.

Remark 2. It should be noticed that for any CES in $\mathcal{H}_{2,n}$ the first assumption of the theorem is always obeyed. In order to see it explicitly, let us write the orthonormal vectors spanning such CES as

$$|\Psi_i\rangle = |0\rangle|\Psi_0^{(i)}\rangle + |1\rangle|\Psi_1^{(i)}\rangle \quad (i = 1, \dots, \dim V) \quad (35)$$

Assume now that in this case the local projection of Π_V is rank deficient for any $|x\rangle \in \mathbb{C}^2$. Taking the vector $|x\rangle = |0\rangle$, the matrix $\Pi_V(x)$ is just a sum of projections onto $|\Psi_0^{(i)}\rangle$. Since here $\Pi_V(x)$ is not of full rank, the vectors $|\Psi_0^{(i)}\rangle$ must be linearly dependent. This, however, immediately implies that there is a product vector in V which contradicts the assumption that V is CES.

Now, let us show that for any three-dimensional CES supported in \mathcal{H}_4 , the first assumption of the theorem is satisfied.

Lemma 1. *Let V be a three-dimensional CES supported in $\mathbb{C}^4 \otimes \mathbb{C}^4$. Then there exist $|x\rangle \in \mathbb{C}^4$ such that the matrix $B(x)$ is of rank 3.*

Proof. Assume in contrary that for any $|x\rangle \in \mathbb{C}^4$ the matrix $B(x)$ is of rank 2 or lower. Notice that rows of $B(x)$ are just vectors $|\Psi_i\rangle$ ($i = 1, 2, 3$) spanning V projected locally on $|x\rangle$ (henceforward denoted by $|\Psi_i(x)\rangle$). Also, in what follows, we denote by $|z'\rangle$ and $|z''\rangle$ the $|z\rangle \in \mathbb{C}^4$ projected onto $\text{span}\{|1\rangle, |2\rangle, |3\rangle\}$ and $\text{span}\{|0\rangle, |2\rangle, |3\rangle\}$, respectively.

Let us start the proof by noting that in $\mathbb{C}^4 \otimes \mathbb{C}^4$ the largest subspace with all vectors having Schmidt rank 4 is one-dimensional (see, e.g., Refs. [21–23]). Therefore, we can assume that one of the vectors spanning our CES V , say $|\Psi_1\rangle$, has Schmidt rank either 2 or 3. As the proof is straightforward but tedious, in what follows we consider only the first case; however, analogous reasoning works also for the case of Schmidt rank 3.

By local unitary operations, we can always bring $|\Psi_1\rangle$ to $|\Psi_1\rangle = |00\rangle + |11\rangle$, while the remaining two vectors take the general forms

$$\begin{aligned} |\Psi_2\rangle &= |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle + |2\rangle|\psi_2\rangle + |3\rangle|\psi_3\rangle \\ |\Psi_3\rangle &= |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle + |3\rangle|\varphi_3\rangle \end{aligned} \quad (36)$$

Our proving strategy is that we consider vectors $|\Psi_i(x)\rangle$ ($i = 1, 2, 3$) for some particular subsets of vectors of \mathbb{C}^4 . Then, demanding that all the 3×3 submatrices of the

3×4 matrix $B(x)$ have zero determinant, we show that V is not either a CES or supported in \mathcal{H}_4 .

We start by noting that the above procedure for $|x\rangle = |0\rangle$ and $|x\rangle = |1\rangle$ leads us to a conclusion that the sets $\{|0\rangle, |\psi_0\rangle, |\varphi_0\rangle\}$ and $\{|1\rangle, |\psi_1\rangle, |\varphi_1\rangle\}$ must be linearly dependent. This means that by taking appropriate linear combinations of $|\Psi_i\rangle$ ($i = 1, 2, 3$), the vectors spanning V can be expressed as

$$\begin{aligned} |\Psi_1\rangle &= |00\rangle + |11\rangle \\ |\Psi_2\rangle &= |0\rangle|\psi_0\rangle + |2\rangle|\psi_2\rangle + |3\rangle|\psi_3\rangle \\ |\Psi_3\rangle &= |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle + |3\rangle|\varphi_3\rangle, \end{aligned} \quad (37)$$

where, in general, vectors $|\psi_i\rangle$ and $|\varphi_i\rangle$ are different than the ones appearing in Eq. (36). However, for convenience, we do not change the notation.

In what follows, we split the proof into two cases, that is, when both vectors $|\psi_0\rangle$ and $|\varphi_1\rangle$ are nonzero and one of them vanishes.

The case of nonzero $|\psi_0\rangle$ and $|\varphi_1\rangle$. By choosing $|x\rangle = (1, \alpha, 0, 0)$ ($\alpha \in \mathbb{C}$) and demanding that the corresponding matrix $B(x)$ is of rank at most 2 (i.e., that all the 3×3 determinants of this matrix vanish), we get the conditions that either $|\psi_0\rangle \parallel |\varphi_1\rangle$ or $|\psi_0\rangle, |\varphi_1\rangle \in \text{span}\{|0\rangle, |1\rangle\}$.

In both cases, we can assume that either $|\psi_0\rangle \not\parallel |0\rangle$ or $|\varphi_1\rangle \not\parallel |1\rangle$ because otherwise one can remove $|\psi_0\rangle$ or $|\varphi_1\rangle$ from the vectors (37) by taking appropriate linear combinations of $|\Psi_i\rangle$ ($i = 1, 2, 3$), leading us to the case considered below. Let us then assume that $|\psi_0\rangle \not\parallel |0\rangle$, which in particular means that $|\psi'_0\rangle \neq 0$.

Now, we project onto $|x\rangle = (1, 0, \alpha, \beta)$ ($\alpha \in \mathbb{C}$), which allows us to conclude that $|\psi'_0\rangle$ has to be parallel to all the vectors of $|\psi'_i\rangle$ and $|\varphi'_i\rangle$ ($i = 2, 3$) which are nonzero, unless $|\varphi'_2\rangle = 0$ and $|\varphi'_3\rangle = 0$. In the former case, one immediately finds that V is not supported in \mathcal{H}_1 . This is because either $|\psi_0\rangle$ is proportional to $|\varphi_1\rangle$, meaning that $|\psi'_0\rangle \parallel |\varphi'_1\rangle \parallel |\psi'_2\rangle \parallel |\varphi'_2\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ and therefore $V \subseteq \mathbb{C}^4 \otimes \text{span}\{|0\rangle, |1\rangle, |\psi'_0\rangle\}$, or when $|\psi_0\rangle, |\varphi_1\rangle \in \text{span}\{|0\rangle, |1\rangle\}$, the vectors $|\psi'_0\rangle$ and $|\psi'_i\rangle$ and $|\varphi'_i\rangle$ ($i = 2, 3$) are parallel and hence $V \subseteq \mathbb{C}^4 \otimes \mathbb{C}^2$.

In the case when $|\varphi'_2\rangle = 0$ and $|\varphi'_3\rangle = 0$, we notice that it cannot happen that $|\varphi_2\rangle = |\varphi_3\rangle = 0$ because in such case there exists a product vector $|1\rangle|\varphi_1\rangle$ in V . Therefore, one of the vectors $|\varphi_2\rangle$ or $|\varphi_3\rangle$ is proportional to $|0\rangle$. We project further onto $|x\rangle = (0, 1, \alpha, \beta)$ ($\alpha, \beta \in \mathbb{C}$), which, in the same way as before, leads us to the additional conditions that $|\psi'_2\rangle \parallel |\psi'_3\rangle|0\rangle$, and so $|\psi_2\rangle, |\psi_3\rangle \in \text{span}\{|0\rangle, |1\rangle\}$. Then all vectors $|\psi_i\rangle$ and $|\varphi_i\rangle$ ($i = 2, 3$) live in $\text{span}\{|0\rangle, |1\rangle\}$. Consequently, irrespectively of whether $|\psi_0\rangle$ is proportional to $|\varphi_1\rangle$ or $|\psi_0\rangle, |\varphi_1\rangle \in \{|0\rangle, |1\rangle\}$, V is not supported in \mathcal{H}_4 .

The case when one of the vectors $|\psi_0\rangle, |\varphi_1\rangle$ vanishes. Without any loss of generality we can assume that $|\varphi_1\rangle = 0$. This immediately implies that one of the vectors $|\varphi'_2\rangle$ or $|\varphi'_3\rangle$ is nonzero because otherwise $|\Psi_3\rangle = |\omega\rangle|0\rangle$ for some $|\omega\rangle \in \mathbb{C}^4$, meaning that there is a product vector in V .

We now project locally $|\Psi_i\rangle$ ($i = 1, 2, 3$) onto $|x\rangle = (1, 0, \alpha, \beta)$ ($\alpha, \beta \in \mathbb{C}$). Demanding that the resulting matrix $B(x)$ is of rank less than or equal to 2, we get the condition that if $|\psi'_0\rangle$ is nonzero the vectors $|\psi'_i\rangle$ ($i = 0, 2, 3$) and $|\varphi'_i\rangle$ ($i = 2, 3$) are parallel (excluding the ones that possibly vanish). As a result, V is a subspace supported on $\mathbb{C}^4 \otimes \text{span}\{|0\rangle, |1\rangle, |\psi'_0\rangle\}$.

If, on the other hand, $|\psi'_0\rangle = 0$ ($|\psi'_0\rangle \parallel |0\rangle$), one gets the conditions that either $|\varphi'_2\rangle = \eta|\psi'_2\rangle$ and $|\varphi'_3\rangle = \eta|\psi'_3\rangle$ or $|\psi'_2\rangle \parallel |\varphi'_2\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ (again excluding the vectors that vanish). In the first of these cases, we can subtract $|\Psi_3\rangle$ from $|\Psi_2\rangle$ obtaining a product vector in V , while in the second case, again V is supported on a subspace $\mathbb{C}^4 \otimes \text{span}\{|0\rangle, |1\rangle, |\psi'_2\rangle\}$, leading to a contradiction with one of the assumptions.

Now let us consider the case when one of the vectors is of Schmidt rank 3 and there is no vector of Schmidt rank 2 in the CES V . The vectors spanning V can be written in this case as

$$\begin{aligned} |\Psi_1\rangle &= |00\rangle + |11\rangle + |22\rangle, \\ |\Psi_2\rangle &= |0\rangle|\psi_0\rangle + |2\rangle|\psi_2\rangle + |3\rangle|\psi_3\rangle, \\ |\Psi_3\rangle &= |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle + |3\rangle|\varphi_3\rangle, \end{aligned} \quad (38)$$

where the sets $\{|\psi_0\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ and $\{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$ are linearly independent (otherwise the Schmidt rank of $|\Psi_2\rangle$ or $|\Psi_3\rangle$ would be less than 3). In particular, none of these six vectors can be zero.

Choosing $|x\rangle = (1, 0, 0, \alpha)$ and $|x\rangle = (0, 1, 0, \alpha)$ ($\alpha \in \mathbb{C}$), one gets that $|\psi'_0\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ or $|\varphi'_3\rangle = 0$ and $|\varphi'_1\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ or $|\psi'_3\rangle = 0$. This gives us four cases: (i) $|\psi'_0\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ and $|\varphi'_1\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$, (ii) $|\psi'_0\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ and $|\psi'_3\rangle = 0$, (iii) $|\varphi'_1\rangle \parallel |\psi'_3\rangle \parallel |\varphi'_3\rangle$ and $|\varphi'_3\rangle = 0$, and (iv) $|\varphi'_3\rangle = 0$ and $|\psi'_3\rangle = 0$ (i.e., $|\varphi_3\rangle \parallel |0\rangle$ and $|\psi_3\rangle \parallel |1\rangle$).

In the first case, we project onto $(1, \alpha, 0, 0)$ ($\alpha \in \mathbb{C}$), which gives us additional conditions that either $|\psi_0\rangle \parallel |\varphi_1\rangle$ or $|\psi_0\rangle, |\varphi_1\rangle \in \text{span}\{|0\rangle, |1\rangle\}$. In the first of these two possibilities, we can additionally assume that $|\psi'_0\rangle \not\parallel |1\rangle$ because otherwise we fall into the second possibility. This implies that in particular $|\psi_0\rangle \parallel |\psi_3\rangle$, meaning that $|\Psi_2\rangle$ is of Schmidt rank 2.

In the case when $|\psi_0\rangle, |\varphi_1\rangle \in \text{span}\{|0\rangle, |1\rangle\}$, we easily find that $|\psi_0\rangle, |\varphi_1\rangle, |\psi_3\rangle, |\varphi_3\rangle \in \text{span}\{|0\rangle, |1\rangle\}$. Projecting further onto $(0, 0, 1, \alpha)$ ($\alpha \in \mathbb{C}$) and after careful analysis, one arrives at the conclusion that again one of the vectors spanning V is of Schmidt rank 2.

Let us now come to the cases (ii) and (iii). As they are analogous, for simplicity we consider only the first of them. The corresponding conditions imply that $|\psi_0\rangle \parallel |1\rangle$ and $|\psi_0\rangle, |\varphi_3\rangle \in \text{span}\{|0\rangle, |1\rangle\}$. Then, it suffices to project onto $|x\rangle = (1, \alpha, 0, 0)$ ($\alpha \in \mathbb{C}$) to see that $|\varphi_1\rangle \in \text{span}\{|0\rangle, |1\rangle\}$, meaning that we fall into the already discussed case when $|\psi_0\rangle, |\varphi_1\rangle, |\psi_3\rangle, |\varphi_3\rangle \in \text{span}\{|0\rangle, |1\rangle\}$ (see above).

Let us consider the case (iv) and project, for instance, onto $|x\rangle = (1, 0, \alpha, 0)$ ($\alpha \in \mathbb{C}$). This, in particular, imposes the conditions that $|\psi'_0\rangle \parallel |\varphi'_2\rangle$ or $|\varphi'_2\rangle = 0$. In

the latter case, $|\varphi_2\rangle \parallel |\varphi_3\rangle \parallel |0\rangle$ and one sees immediately that $|\Psi_3\rangle$ is of Schmidt rank at most 2. Assume then that $|\psi'_0\rangle \parallel |\varphi'_2\rangle$ and recall that either $|\psi_0\rangle$ and $|\varphi_1\rangle$ are parallel or they both live in $\text{span}\{|0\rangle, |1\rangle\}$. This implies that either $|\varphi'_1\rangle \parallel |\varphi'_2\rangle$ or $|\varphi_1\rangle, |\varphi_2\rangle \in \text{span}\{|0\rangle, |1\rangle\}$. In the first case, $|\Psi_3\rangle \in \mathbb{C}^4 \otimes \text{span}\{|0\rangle, \varphi'_1\rangle\}$, while in the second one $|\Psi_3\rangle \in \mathbb{C}^4 \otimes \text{span}\{|0\rangle, |1\rangle\}$. Hence, both cases are of rank at most 2. This completes the proof. \square

Finally, let us follow Remark 1 and prove that any CES in \mathcal{H}_n of dimension less than $n - 1$ is a subspace of the $(n - 1)$ -dimensional CES. For this purpose, we prove a bit more general fact.

Lemma 2. *Any r -dimensional CES in $\mathcal{H}_{m,n}$ with $r < (m + 1)(n + 1)$ is a subspace of an $(r + 1)$ -dimensional CES.*

Proof. Let $V \subset \mathcal{H}_{m,n}$ denote an r -dimensional CES and M_V denote a set constructed from linear combinations of vectors from V and all product vectors from $\mathcal{H}_{m,n}$, that is,

$$M_V = \{|\Psi\rangle + |e\rangle \otimes |f\rangle \mid |\Psi\rangle \in V, |e\rangle \in \mathbb{C}^n, |f\rangle \in \mathbb{C}^m\}. \quad (39)$$

We start by noting that V can be extended to some $(r + 1)$ -dimensional CES, denoted V_2 , if $M_V \subsetneq \mathcal{H}_{m,n}$. This is because if the set M_V is a proper subset of $\mathcal{H}_{m,n}$, we can take a vector $|\Psi\rangle \in \mathcal{H}_{m,n}$ which does not belong to M_V and define the subspace V_2 as a linear hull of that vector and V . The construction of M_V guarantees that V_2 is CES because otherwise $|\Psi\rangle$ can be written as a linear combination of a vector from V and a product vector from V_2 . Consequently, $|\Psi\rangle$ has to be in M_V , contradicting the assumption that $|\Psi\rangle \notin M_V$.

Let us now show that for any CES of nonmaximal dimension $[\dim V < (n - 1)(m - 1)]$ the set M_V is not the whole $\mathbb{C}^n \otimes \mathbb{C}^m$. For this purpose, observe first that M_V is a manifold. Every its element can be represented as a sum of an element from $(n + m - 1)$ -dimensional manifold of separable vectors from $\mathcal{H}_{m,n}$ and the r -dimensional CES V , and hence the dimension of M_V is less or equal to $r + n + m - 1$. Therefore, for any $r < (m - 1)(n - 1)$, the dimension of M_V is less than mn , meaning that $M_V \subsetneq \mathcal{H}_{m,n}$. Consequently, V can be extended to an $(r + 1)$ -dimensional CES. \square

By repeating the above procedure until $r = (m - 1)(n - 1)$ we see that every CES of a nonmaximal dimension in $\mathcal{H}_{m,n}$ is a subspace of some maximally dimensional CES.

In conclusion, we see, via theorem 1 and both the above lemmas, that all completely entangled subspaces of \mathcal{H}_4 of dimension less than or equal to 3 have the property of spanning. This, by virtue of the previous discussion, proves the following theorem.

Theorem 2. *Let W be a decomposable witness acting on \mathcal{H}_4 . Then, the following statements are equivalent:*

(i): $W = Q^\Gamma$, $Q \geq 0$, $r(Q) \leq 3$, and $\text{supp}(Q)$ is a CES supported in \mathcal{H}_4 ,

(ii): P_W spans \mathcal{H}_4 ,

(iii): W is optimal.

Collecting together results obtained for \mathcal{H}_2 and \mathcal{H}_3 in Ref. [14] and the above one for \mathcal{H}_4 , it is tempting to conjecture that any CES V such that $\dim V \leq n - 1$ supported in \mathcal{H}_n has the property of spanning. In other words, we conjecture that Theorem 2 holds for any DEW (5) from \mathbb{M}_n as long as $r(Q) \leq n - 1$.

Interestingly, it is easy to disprove this conjecture for systems of unequal local dimensions. For this purpose, let us consider the subspace of $\mathcal{H}_{m,n}$ with $n > m$, spanned by the following vectors:

$$\begin{aligned} |\Psi_i\rangle &= |0\rangle|i\rangle - |i\rangle|0\rangle & (i = 1, \dots, m - 1) \\ |\Psi_i\rangle &= |0\rangle|i\rangle - |1\rangle|i - 1\rangle & (i = m, \dots, n - 2) \\ |\Psi_{n-1}\rangle &= |0\rangle|n - 1\rangle - |\psi_{\text{ant}}\rangle, \end{aligned} \quad (40)$$

where $|\psi_{\text{ant}}\rangle$ is a state from the antisymmetric subspace of $\mathbb{C}^m \otimes \mathbb{C}^m$ orthogonal to all $|\Psi_i\rangle$ ($i = 1, \dots, m - 1$). Notice that in the case $n = m + 1$, one omits the second group of vectors.

Using the matrix representation of the above vectors and applying analogous reasoning as in the Appendix, one finds that $V = \text{span}\{|\Psi_i\rangle\}_{i=1}^{n-1}$ is a $(n - 1)$ -dimensional subspace supported in $\mathcal{H}_{m,n}$ which does not contain any product vectors. Then, there are two classes of product vectors orthogonal to this subspace, that is,

$$(1, x_1, \dots, x_{m-1}) \otimes (y_0, \dots, y_{n-2}, 0) \quad (41)$$

and

$$(0, x_1, \dots, x_{m-1}) \otimes |y\rangle, \quad (42)$$

where now $|y\rangle$ can be arbitrary. Let us now take partial conjugations of both classes. Clearly, the product vector $|0\rangle|n - 1\rangle$ is orthogonal to both of them and therefore the above subspace does not have the property of spanning. To illustrate the above construction with an example, let us consider the case of $\mathcal{H}_{3,4}$. The above construction gives $\mathcal{V} = \text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |03\rangle - (|12\rangle - |21\rangle)\}$. The product vectors orthogonal to \mathcal{V} are given by $(1, x_1, x_2) \otimes (y_0, y_1, y_2, 0)$ ($x_1, x_2, y_0, y_1, y_2 \in \mathbb{C}$) and $(0, x_1, x_2) \otimes |y\rangle$ ($x_1, x_2 \in \mathbb{C}, |y\rangle \in \mathbb{C}^4$). The vector orthogonal to partial conjugations of both classes is $|0\rangle|3\rangle$.

V. CONCLUSION

Let us here shortly summarize the obtained results and outline the possibilities for further research.

Because entanglement witnesses are very useful tools in quantum information theory, their characterization is of great interest. Particularly important in this context is the notion of optimality introduced in Ref. [10].

Very recently, some of us have studied decomposable witnesses detecting entanglement in qubit-qunit systems and provided a complete characterization of optimality

in this case [14]. In the present paper we have treated several questions that arise in other finite-dimensional Hilbert spaces. One of the most interesting problems is whether optimal decomposable witnesses exist such that the corresponding \mathcal{P}_W s do not span the Hilbert space. Here we answer this question positively by showing that for any Hilbert space $\mathcal{H}_{m,n}$ with $m, n \geq 3$ there exist optimal DEWs without the property of spanning.

Aiming at the generalization of the results of Ref. [14], we have tried to distinguish all the CESs for which the corresponding \mathcal{P}_V s do span $\mathcal{H}_{m,n}$. This tells us for which DEWs of form (5) the implications (i) and (ii) (see Sec. II) become equivalences. We have proven that under a certain condition, which is conjectured to be always satisfied, all $(n-1)$ -dimensional CESs supported in \mathcal{H}_n have the property of spanning. This result obviously extends to any CES being a proper subspace of such an $(n-1)$ -dimensional CES supported in two-qubit Hilbert space. We have proven that any r -dimensional CES can be extended to an $(r+1)$ -dimensional one, meaning that all CESs in \mathcal{H}_n of a dimension less than $n-1$ are subspaces of $(n-1)$ -dimensional CESs. We have applied this statements to the two-ququart case and shown that any CES of dimension $\dim V \leq 3$ supported in \mathcal{H}_4 has the property of spanning. This, together with the results obtained in Ref. [14], allows us to conjecture that any CES of dimension less than or equal to $n-1$ and supported in \mathcal{H}_n has this property. Certainly this conjecture cannot hold if the local dimensions are different, as we provide examples of $(n-1)$ -dimensional subspaces without the property of spanning supported in $\mathcal{H}_{m,n}$ ($3 \leq m < n$).

Clearly, still much has to be done to complete the characterization of witnesses. In particular, it remains to prove the above conjecture and determine whether all CESs of $\mathcal{H}_{m,n}$ ($m, n \geq 3$) have the property of spanning. This would allow to find all the instances when (i) and (ii) (see Sec. II) are equivalences. The more challenging task, however, would be to ask a similar question in the case of indecomposable witnesses.

ACKNOWLEDGMENTS

Discussions with Seung-Hyeok Kye, Paweł Mazurek, Łukasz Skowronek, and Julia Stasińska are gratefully acknowledged. This work was supported by EU IP AQUATE, Spanish MINCIN Project FIS2008-00784 (TOQATA), Consolider Ingenio 2010 QOIT, EU STREP Project NAMEQUAM, and the Alexander von Humboldt Foundation. M. L. acknowledges NFS Grant No. PHY005-51164.

Dodatek A: Proving that considered subspaces are CES

Let us first recall the vector-matrix isomorphism. Precisely, to any vector $\mathcal{H}_{m,n} \ni |\psi\rangle = \sum_{ij} a_{ij} |ij\rangle$ there cor-

respond a complex $m \times n$ matrix $A = \sum_{ij} a_{ij} |i\rangle\langle j|$ (and vice versa).

1. The symmetric case

We consider an m -dimensional subspace V in $\mathbb{C}^m \otimes \mathbb{C}^m$ spanned by vectors:

$$|\Psi_i\rangle = |0\rangle|i\rangle - |i\rangle|0\rangle \quad (i = 1, \dots, m-1),$$

and a vector $|\Psi_m\rangle \in \text{span}(|1\rangle, \dots, |m-1\rangle)^{\otimes 2}$.

We prove that the only product vector in this subspace is zero. For this purpose, let us take a linear combination of all vectors spanning V , which, in terms of the above vector-matrix correspondence, reads

$$\begin{bmatrix} 0 & \alpha_1 & \dots & \alpha_{m-1} \\ -\alpha_1 & & & \\ \vdots & & \alpha_m \tilde{A}_m & \\ -\alpha_{m-1} & & & \end{bmatrix} \quad (\text{A1})$$

where $r(A_m) > 1$. There exists a product vector in V iff there exist nonzero α s such that the above matrix is of rank 1.

First observe that $|\Psi_m\rangle$ affects neither the first column nor the first row of (A1). The principal minors of the matrix (A1) formed by the sets of indices $\{0, i\}$, $i \in \{1, m-1\}$ are then equal $\alpha_1^2, \dots, \alpha_{m-1}^2$. If the matrix is to be of rank 1, all of them have to be zero, which implies that $\alpha_1 = \dots = \alpha_{m-1} = 0$. The last coefficient α_m has also to be zero, because $r(A_m) > 1$. This finishes the proof.

2. The general case

Now we consider an n -dimensional subspace in $\mathbb{C}^m \otimes \mathbb{C}^n$ spanned by vectors $|\Psi_i\rangle$ ($i = 1, \dots, n-1$) given by Eqs. (7) and (18), but in this case the vector $|\Psi_m\rangle$ is fixed and defined by (15). The combination matrix has the form

$$\begin{bmatrix} 0 & \alpha_1 & \dots & \alpha_{m-1} & \alpha_{m+1} & \alpha_{m+2} & \dots & \alpha_n \\ -\alpha_1 & & & x & -\alpha_{m+2} & \dots & \alpha_n & 0 \\ \vdots & & \alpha_m A_m & & 0 & \dots & \dots & 0 \\ -\alpha_{m-1} & & & & 0 & \dots & \dots & 0 \end{bmatrix}, \quad (\text{A2})$$

where $x = \alpha_m a_{0,m-2} - \alpha_{m+1}$.

Again, by calculating the principal minors of the combination matrix formed by the sets of indices $\{0, i\}$, $i \in \{1, m-1\}$, one gets $\alpha_0 = \dots = \alpha_{m-2} = 0$. The block formed by the last $m-1$ rows and the columns of indices $1, \dots, m-1$ is affected by $|\Psi_m\rangle$ and $|\Psi_{m+1}\rangle$, but always has rank $m-1$ if only $\alpha_m \neq 0$, so $\alpha_m = 0$. Now only the two first rows are nonzero in the combination matrix. We consider minors of size 2 formed by these rows and

columns of indices $i - 1$, i . Starting with $i = m + 1$, one gets $\alpha_{m+1}^2 = 0$. Taking $i = m + 2$, one gets $\alpha_{m+2}^2 = 0$. In

this manner, one can show that $\alpha_{m+1} = \dots = \alpha_n = 0$, completing the proof.

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